An Iterative Method for Algebraic Solution to Interval Equations

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Abstract

The algebraic solution to systems of linear equations involving an interval square matrix and an interval right-hand side vector in terms of interval arithmetic is discussed. We formulate an iterative method and prove its convergence, under certain conditions on the interval matrix. In the special case when only the right-hand side is interval we reduce the problem to two ordinary linear systems and their solution is used to provide initial approximation for the iterative solution of the general problem. An iterative numerical algorithm is proposed.

Key words: iterative methods, interval linear system

1 Introduction

A linear algebraic system $\mathbf{A}x = \mathbf{b}$ involving intervals in the $(n \times n)$-matrix $\mathbf{A}$ and in the right-hand side n-vector $\mathbf{b}$, relates to several different problems, resp. solution sets (see e. g. the classification in [18]). Here we shall be concerned with the *interval algebraic solution* which is an interval vector $\mathbf{x}$ satisfying the system whenever the arithmetic operations are performed in interval arithmetic. We shall write the problem in the form

$$\mathbf{A} \times \mathbf{x} = \mathbf{b}, \tag{1}$$

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in order to emphasize that the symbol "$\times$" means *interval multiplication* (in the sense of [1], [2], [13]) and that the solution vector $\mathbf{x}$ is generally an interval vector.

We use for the interval arithmetic suitable notation, which is convenient to formulate new ralation and to perform symbolic transformations. Using symbolic notations based on binary variables taking values from the set $\{+, -\}$, one can present a generalized interval (which is a pair of real numbers) as a complex of two elements: i) a proper (normal) interval and ii) a binary variable called *type* or *direction* [7]. In terms of these concepts and notations we have formulated new distributivity relations and simple rules for the symbolic transformation of algebraic expressions and equations [5]–[7]. Using such approach the interval algebraic structure, further called *directed interval algebraic (arithmetic) system*, gains new powerful features and incorporates both Kaucher's extended interval arithmetic and the extended arithmetic for normal intervals using *inner* (*nonstandard*) operations developed earlier by the author [5]. In particular it has been proved that: i) extended interval arithmetic with inner operations is a "projection" of the directed interval arithmetic system on the system of normal intervals, and ii) extended interval arithmetic can be isomorphically embedded into directed interval algebraic system [7], [8]. Directed interval arithmetic provides a simple general platform which incorporates several known interval systems and enables us to pass from using directed (proper and improper) intervals to using only proper intervals (with inner operations), and vice versa. Briefly, directed interval arithmetic is the interval arithmetic described in [1], [2], [13]), which: 1) due to the use of special symbolic "plus-minus" notations, incorporates the exended interval arithmetic using inner operations, and 2) is equipped by a set of new relations (formulated in terms of the "plus-minus" notations), necessary for the straightforward symbolic transformation of algebraic expressions and equations.

It has been noticed that the interval algebraic solution (1) is closely related to the solutions of several practical linear algebraic problems involving interval coefficients, such as the united, the controlled and the tolerable solutions [1]–[3], [16]–[18]. This shows the importance of the study of the algebraic solution to (1).

In this work we propose an interval iteration procedure for finding the algebraic solution of the interval system (1) and prove its convergence using new powerfull tools from directed interval arithmetic. Our iteration procedure essentially uses some of the methods proposed in [3], [20], but is formulated in simple form. A numerical algorithm is also formulated. Part of this work has been presented in [8], [11]. We first introduce the concepts of directed interval arithmetic, necessary to define the solution of (1).

In the special case when only the right-hand side is interval we reduce the problem to two ordinary linear systems. The solution of this special problem is used to provide initial approximation for the iterative solution of the general problem.

2 Directed Interval Arithmetic

Denote by $D = \{[\alpha, \beta] \mid \alpha, \beta \in \mathbb{R}\}$ the set of ordered couples of real numbers. We call the elements of D *directed intervals*. The first component of $a \in D$ is further denoted by a^-, and the second by a^+, so that $a = [a^-, a^+]$. Thus $a^\lambda \in \mathbb{R}$ with $\lambda \in \Lambda = \{+, -\}$ is the first or second component of $a \in D$ depending on the value of λ. The binary variable λ may be expressed as a "product" of two other binary variables, $\lambda = \mu\nu$, $\mu, \nu \in \Lambda$, we assume that $++ = -- = +$, and $+- = -+ = -$. The directed interval $a = [a^-, a^+]$ is called *proper* (*normal*), if $a^- \leq a^+$, *degenerated* if $a^- = a^+$ (in this case we write $a = [a, a]$), and *improper* if $a^- > a^+$. The set of all proper intervals is denoted by $I(\mathbb{R})$, the set of degenerated intervals is equivalent to $\mathbb{R}$ and is denoted again by $\mathbb{R}$, and the set of improper intervals is denoted by $\overline{I(\mathbb{R})}$. We have $D = I(\mathbb{R}) \cup \overline{I(\mathbb{R})}$.

To every directed interval $a = [a^-, a^+] \in D$ corresponds a binary variable *type* or *direction*, defined by

$$\tau(a) = \begin{cases} +, \text{ if } \ a^- \leq a^+; \\ -, \text{ if } \ a^- > a^+, \end{cases}$$

and a *proper interval*

$$\operatorname{pro}(a) = \begin{cases} [a^-, a^+], \text{ if } \ \tau(a) = +; \\ [a^+, a^-], \text{ if } \ \tau(a) = -. \end{cases}$$

We have $\operatorname{pro}(a) = [a^{-\tau(a)}, a^{\tau(a)}] \in I(\mathbb{R})$, due to $a^{-\tau(a)} \leq a^{\tau(a)}$. The functionals pro and τ provide a simple relation between propositions formulated in terms of directed intervals and one's made in terms of proper (normal) intervals [7].

We shall use the following notation

$$Z = \{a \in I(\mathbb{R}) \mid a^- \leq 0 \leq a^+\},$$

$$Z^* = \{a \in I(\mathbb{R}) \mid a^- < 0 < a^+\},$$

$$\overline{Z} = \{a \in \overline{I(\mathbb{R})} \mid a^+ \leq 0 \leq a^-\},$$

$$\overline{Z}^* = \{a \in \overline{I(\mathbb{R})} \mid a^+ < 0 < a^-\},$$

$$\mathcal{T} = Z \cup \overline{Z}, \qquad \mathcal{T}^* = Z^* \cup \overline{Z}^*, \qquad D^* = D \setminus \mathcal{T}.$$

In D we define the functional *sign* of a directed interval $\sigma : D \to \Lambda$ by

$$\sigma(a) = \begin{cases} +, \text{ if } \ a^- + a^+ \geq 0; \\ -, \text{ if } \ a^- + a^+ > 0. \end{cases}$$

Obviously, $\sigma(a) = \sigma(\mathrm{pro}(a))$.

The operation addition "+" is defined in D by:

$$a + b = [a^- + b^-, a^+ + b^+], \quad a, b \in D. \tag{2}$$

The definition of multiplication "×" in D is rather complex. The following expressions can be used

$$a \times b = \begin{cases} [a^{-\sigma(b)} b^{-\sigma(a)},\ a^{\sigma(b)} b^{\sigma(a)}], \quad a, b \in D \setminus \mathcal{T}, \\ [a^{\delta\tau(b)} b^{-\delta},\ a^{\delta\tau(b)} b^{\delta}], \quad \delta = \sigma(a), \quad a \in D \setminus \mathcal{T},\ b \in \mathcal{T}^*, \\ [a^{-\delta} b^{\delta\tau(a)},\ a^{\delta} b^{\delta\tau(a)}], \quad \delta = \sigma(b), \quad a \in \mathcal{T}^*,\ b \in D \setminus \mathcal{T}. \end{cases} \tag{3}$$

$$a \times b = \begin{cases} [\min\{a^- b^+, a^+ b^-\},\ \max\{a^- b^-, a^+ b^+\}], \quad a, b \in Z^*, \\ [\max\{a^- b^-, a^+ b^+\},\ \min\{a^- b^+, a^+ b^-\}], \quad a, b \in \overline{Z}^*, \\ 0, \quad (a \in Z^*,\ b \in \overline{Z}^*) \ \vee \ (a \in \overline{Z}^*,\ b \in Z^*). \end{cases} \tag{4}$$

Formulae (3), (4) present component-wise the product of two directed intervals and produce same results as in [2], [1], [13]).

Formula (3) shows that $a \times b$ is defined by two end-point multiplications in case that at least one of the intervals is from $D \setminus \mathcal{T}$. Formula (4) refers to the case $a, b \in \mathcal{T}^*$ and uses four end-point multiplications for the case $\tau(a) = \tau(b) = \tau$ (for $\tau(a) = -\tau(b)$ the result is 0).

Recall that addition and multiplication for normal intervals $[a, b], [c, d] \in I(\mathbb{R})$ are defined by $[a, b] \star [c, d] = \{x \star y \mid a \leq x \leq b, c \leq y \leq d\}$, $\star \in \{+, \times\}$ [19].

The restrictions of the expressions (2), (3)–(4) on $I(\mathbb{R})$ produce these familiar results for normal intervals. To say it differently (2), (3)–(4) are obtained by isomorphic embedding of the semigroups $(I(\mathbb{R}), +)$, resp. $I(\mathbb{R}), \times)$, into (minimal) groups. All operations in directed interval arithmetic can be deduced from the basic operations for addition and multiplication (of normal intervals) by means of familiar algebraic constructions (like embeddings, inverses, cxompositions, projections, etc.).

We shall (sometimes) replace the symbol "$\times$" by "$*$" to denote the simple case when one of the multipliers in a product $a \times b$ is degenerate, writing thus $a * b$ or $\alpha * b$ (multiplication by scalar). From (3) for $[a, a] = a \in \mathbb{R}$, $b \in D$ we have $a * b = [ab^{-\sigma(a)}, ab^{\sigma(a)}]$. The operator *negation* (negative element) is defined by $\mathrm{neg}(b) = \neg b = (-1) * b = [-b^+, -b^-]$. The restriction of the composite operation $a + (-1) * b = a + (\neg b) = a \neg b = [a^- - b^+, a^+ - b^-]$ for $a, b \in I(\mathbb{R})$ is the familiar subtraction of intervals.

Remark. In the literature on interval analysis and set-valued (convex) analysis the operator $(-1) * b$ is denoted traditionally by $-b$, and $a + (-1) * b$ — by $a + (-b) = a - b$. Further, we shall also keep to this familiar notation. However we warn the reader to be careful when seeing the sign "$-$" in front of an interval, due to the different properties of negation in comparisson to opposite, e. g. the relation $a - a = 0$ is not valid for nondegenerate intervals. We recommend the reader to replace (by pencil) the "$-$" in front of all intervals by the symbol $\neg$ (and to do similar marks with the symbol / to appear later.

The groups $(D, +)$ and $(D^*, \times)$ possess inverse additive and multiplicative operators. Using this operators and negation we introduce additional operators as follows.

Denote by $\mathrm{opp}(a) = -_D a$ the opposite (additive inverse) element of $a \in D$ and by $\mathrm{inv}(a) = 1/_D a$ the inverse element of $a \in D^*$ with respect to "$\times$". For the inverse elements we have the component-wise presentations $-_D a = [-a^-, -a^+]$, for $a \in D$, and $1/_D a = [1/a^-, 1/a^+]$, for $a \in D^*$. Using the group properties of $(D, +)$, $(D^*, \times)$ we can solve algebraic equations. For $a, b \in D$ the unique solution to the equation $a + x = b$ is $x = b + (-_D a)$. Similarly, for $a \in D^*, b \in D$ the unique solution to the equation $a \times y = b$ is $y = b \times (1/_D a)$.

The operator *dual element* is defined as composition of negation and opposite by $\mathrm{dual}(a) = a_- = [a^+, a^-]$. The negative, the opposite and the dual elements are related in the following way:

$$a_- = -_D(\neg a) = \neg(-_D a). \tag{5}$$

The operator *reciprocal element*, defined in D^* as $\mathrm{rec}(a) = 1/a = 1/_D a_- = [1/a^+, 1/a^-]$, satisfies

$$1/_D(1/a) = 1/(1/_D a) = a_-. \tag{6}$$

The operation division $a \times (1/b)$ for $a \in D$, $b \in D^*$, is denoted by a/b. From (5) and (6) we obtain for the inverse operators w. r. t. addition and multiplication $-_D a = -a_-$, resp., $1/_D a = 1/a_-$. The inverse elements $-_D a$, $1/_D a$ generate the *inverse operations* $a -_D b = a + (-_D b) = a + (-b_-) = a - b_-$, $a/_D b = a \times (1/_D b) = a \times (1/b_-) = a/b_-$. The (unique) solutions of the equations $a + x = b$, $a \times y = b$, $a + c \times z = b$ are $x = b + (-a_-) = b - a_-$, resp. $y = b \times (-a_-) = b/a_-$, $z = (b - a_-)/c_-$.

To summarize, the algebraic system $(D, +, \times)$ involves the operations subtraction $a - b$, division a/b, the operator dual element a_-, and the operations $a - b_-$, a/b_-, $a_- - b$, a_-/b. Similarly, we can compose $a + b_-$, $a \times b_-$, $a_- + b$, $a_- \times b$ etc.

Denote $a_+ = a$, then we may write $a_\lambda \in \{a, a_-\}$ with $\lambda = \{+, -\}$. The latter notation allows a simple formulation of the distributivity relation in $D \setminus \mathcal{T}$, called *conditionally distributive* or *pseudodistributive law*, see [7]–[9], [12]:

Proposition 1. (Conditionally Distributive Law) For $a, b, c, a + b \in D \setminus \mathcal{T}$ we have

$$(a + b) \times c_{\sigma(a+b)} = (a \times c_{\sigma(a)}) + (b \times c_{\sigma(b)}). \tag{7}$$

More distributive relations can be found in [10].

3 Interval Matrix Algebra

Operations between matrices of directed intervals are defined similarly to matrix operations involving normal intervals. Sum (difference) of two interval matrices of identical size is an interval matrix of the same size formed by component-wise sums (differences). If $\mathbf{A} = (a_{ij}) \in D^{m \times l}$ and $\mathbf{B} = (b_{ij}) \in D^{l \times n}$, then the product of the directed interval matrices $\mathbf{A}$ and $\mathbf{B}$ is the matrix $\mathbf{C} = (c_{ij}) \in \mathbb{R}^{m \times n}$ with $c_{ij} = \sum_{k=1}^{l} a_{ik} \times b_{kj}$. This defines the expression $\mathbf{A} \times \mathbf{x}$ in (1) as a product of two interval matrices: namely, $\mathbf{A}$, $\mathbf{x}$ are directed interval matrices of order $n \times n$, $n \times 1$ resp. and the result $\mathbf{A} \times \mathbf{x}$ is a $n \times 1$ interval matrix.

As a norm in D we take $\| \mathbf{x} \| = \max\{| x^- |, | x^+ |\}$ [2]. The norm is defined for vectors and matrices in the usual way. For instance, for $\mathbf{A} = (a_{ik}) \in D^{n \times n}$,

$\| \mathbf{A} \| = \max_i \{\sum_{k=1}^n \| a_{ik} \|\}$. For the product of two interval matrices we have $\| \mathbf{A} \times \mathbf{B} \| \leq \| \mathbf{A} \| \| \mathbf{B} \|$. A metric in D^n is defined by $\| \mathbf{x} -_D \mathbf{y} \| = \| \mathbf{x} - \mathbf{y}_- \|$ for $\mathbf{x}, \mathbf{y} \in D^n$.

Proposition 2. For $\mathbf{a}, \mathbf{b}, \mathbf{c} \in D^n$, $\| (\mathbf{c} \times \mathbf{a}) -_D (\mathbf{c} \times \mathbf{b}) \| \leq \| \mathbf{c} \| \| \mathbf{a} -_D \mathbf{b} \|$.

The proof can be obtained by means of (7).

The following fixed-point theorem is a generalization of the fixed-point theorem given in [4].

Proposition 3. Let $\mathbf{U} : D_1 \to D_1$, $D_1 \subseteq D^n$, be a contraction mapping in the sense that there exists $q \in R$, $0 < q < 1$, such that $\| \mathbf{U}(\mathbf{x}) -_D \mathbf{U}(\mathbf{y}) \| < q \| \mathbf{x} -_D \mathbf{y} \|$, for every $\mathbf{x}, \mathbf{y} \in D_1$. Then $\mathbf{U}$ possesses a fixed-point $\mathbf{x}^* \in D_1$ which is the limit of the sequence $\mathbf{x}^{(l+1)} = \mathbf{U}(\mathbf{x}^{(l)})$, $l = 0, 1, \ldots$, with any $\mathbf{x}^{(0)} \in D_1$.

The proof follows the classical proof using properties of "$-_D$" such as $\mathbf{a} -_D \mathbf{a} = 0$ and $(\mathbf{a} -_D \mathbf{b}) + (\mathbf{b} -_D \mathbf{c}) = \mathbf{a} -_D \mathbf{c}$.

4 An Iterative Interval Method

For $\mathbf{A} \in D^{n \times n}$ we denote $\mathbf{D}(\mathbf{A}) = (\mathbf{d}_{ij})$ with $\mathbf{d}_{ii} = \mathbf{a}_{ii}$, $\mathbf{d}_{ij} = 0, i \neq j$. For $\mathbf{D} = \mathbf{D}(\mathbf{A})$ denote $\mathbf{D}^{-1} = (\mathbf{d}_{ij}^*)$, $\mathbf{d}_{ij}^* = 1/\mathbf{a}_{ii-}$, $\mathbf{d}_{ij}^* = 0$, $i \neq j$. Clearly $\mathbf{D}^{-1} \times \mathbf{D} = 1$. In [3] the following iteration method has been proposed

$$\mathbf{x}_i := (\mathbf{b}_i -_D \sum_{j=1, j \neq i}^{n} \mathbf{a}_{ij} \times \mathbf{x}_j)/\mathbf{a}_{ii-}, \quad i = 1, \ldots, n. \tag{8}$$

It has been proved [3] that the iterative process (8) converges to the solution to (1) under special restrictions on the input data **A, b** given in implicit form and a rather restrictive choice of the initial approximation. We formulate explicit conditions on the matrix $\mathbf{A}$ and prove that under these conditions (8) converges to the solution to (1) with an arbitrary initial approximation and arbitrary right-hand side $\mathbf{b}$. We first rewrite (8) in matrix form using interval martix arithmetic:

$$\mathbf{x} := \mathbf{D}^{-1} \times (\mathbf{b} -_D (\mathbf{A} -_D \mathbf{D}) \times \mathbf{x}), \quad \mathbf{D} = \mathbf{D}(\mathbf{A}). \tag{9}$$

Proposition 4. If $\| \mathbf{D}(\mathbf{A})^{-1} \| < q < 1$, $\| \mathbf{A} -_D \mathbf{D}(\mathbf{A}) \| < q < 1$, then (1) has a solution $\mathbf{x}^* \in D^n$ and method (9) converges to $\mathbf{x}^*$ for any $\mathbf{b} \in D^n$ and any initial approximation $\mathbf{x}^{(0)} \in D^n$.

Proof. For $\mathbf{x} \in D^n$ denote $\mathbf{B}(\mathbf{x}) = \mathbf{D}^{-1} \times (\mathbf{b} -_D (\mathbf{A} -_D \mathbf{D}) \times \mathbf{x})$. For **x, y**$\in D^n$ we have

$$
\begin{aligned}
&\| \mathbf{B}(\mathbf{x}) -_D \mathbf{B}(\mathbf{y}) \| \\
&=\| \mathbf{D}^{-1} \times (\mathbf{b} -_D (\mathbf{A} -_D \mathbf{D}) \times \mathbf{x}) -_D \mathbf{D}^{-1} \times (\mathbf{b} -_D (\mathbf{A} -_D \mathbf{D}) \times \mathbf{y}) \| \\
&\leq\| \mathbf{D}^{-1} \|\| (\mathbf{b} -_D (\mathbf{A} -_D \mathbf{D}) \times \mathbf{x}) -_D (\mathbf{b} -_D (\mathbf{A} -_D \mathbf{D}) \times \mathbf{y}) \| \\
&=\| \mathbf{D}^{-1} \|\| (\mathbf{A} -_D \mathbf{D}) \times \mathbf{y} -_D (\mathbf{A} -_D \mathbf{D}) \times \mathbf{x} \| \\
&\leq\| \mathbf{D}^{-1} \|\| \mathbf{A} -_D \mathbf{D} \|\| \mathbf{y} -_D \mathbf{x} \|< q^2 \| \mathbf{y} -_D \mathbf{x} \|,
\end{aligned}
$$

using Proposition 2. The inequality $\| \mathbf{B}(\mathbf{x}) -_D \mathbf{B}(\mathbf{y}) \|< q^2 \| \mathbf{y} -_D \mathbf{x} \|$ shows that $\mathbf{B}$ is a contraction mapping. This combined with Proposition 3 proves the theorem. □

5 Special Case: Interval Right-hand Side

Proposition 5. Let $\mathbf{A} = (a_{i,k}) \in R^{n\times n}$ be a real matrix and let the numbers $a_{i,k}\Delta_{i,k}$, where $\Delta_{i,k}$ is the subdeterminant of $a_{i,k}$, have constant signs for all $i, k = 1, 2, ..., n$. Then for the solution to $\mathbf{A} * \mathbf{x} = \mathbf{b}$ the following Cramer-type formula holds:

$$
(\mathbf{x}_i)_{\sigma(\Delta)} = \frac{1}{\Delta}\sum_{i=1}^{n}(-1)^{i+k}\Delta_{ik}(\mathbf{b}_i)_{\lambda_{i,k}} \overset{Def}{=} \frac{1}{\Delta}\begin{vmatrix} a_{11} & \dots & \mathbf{b}_1 & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{1n} & \dots & \mathbf{b}_n & \dots & a_{nn} \end{vmatrix}, \tag{10}
$$

where $\lambda_{i,k} = (-)^{i+k} = \{+,\ i + k \text{ even};\ -,\ i + k \text{ odd}\}$.

The proof is obtained using the properties of directed intervals. A class of matrices satisfying the conditions of the theorem is the class of Wandermond matrices, appearing in interpolation theory. This makes the above formula suitable for the exact solution of identification problems in an interval interpolation setting [6]. If the conditions of the theorem do not hold, formula (10) can still be used for the computation of an initial approximation to the solution as proposed below.

To demonstrate in more detail a practical application consider a system of equations for the unknowns x_i, $i = 1, \dots, m$, of the form

$$
\begin{aligned}
&\alpha_{11} * x_1 + \alpha_{12} * x_2 + \cdots + \alpha_{1m} * x_m = b_1, \\
&\quad . \quad . \quad . \\
&\alpha_{m1} * x_1 + \alpha_{m2} * x_2 + \cdots + \alpha_{mm} * x_m = b_m,
\end{aligned} \tag{11}
$$

where $\alpha_{ij} \in \mathbb{R}$, $b_i \in D$. Using the centre-radius presentation $x_i = (\xi_i, \eta_i)$, $b_i = (\beta_i, \gamma_i)$, where the first coordinates are n-dimensional vectors and the second are real numbers, we obtain that (11) reduces to two linear systems for the coordinates of the unknowns. For the coordinates $\xi_i \in \mathbb{R}^n$ we have the linear system

$$\begin{aligned} &\alpha_{11} \cdot \xi_1 + \alpha_{12} \cdot \xi_2 + \cdots + \alpha_{1m} \cdot \xi_m = \beta_1, \\ &\quad . \quad . \quad . \\ &\alpha_{m1} \cdot \xi_1 + \alpha_{m2} \cdot \xi_2 + \cdots + \alpha_{mm} \cdot \xi_m = \beta_m. \end{aligned} \tag{12}$$

For the coordinates $\eta_i \in \mathbb{R}$ we obtain the linear system:

$$\begin{aligned} &|\alpha|_{11} \cdot \eta_1 + |\alpha|_{12} \cdot \eta_2 + \cdots + |\alpha|_{1m} \cdot \eta_m = \gamma_1, \\ &\quad . \quad . \quad . \\ &|\alpha|_{m1} \cdot \eta_1 + |\alpha|_{m2} \cdot \eta_2 + \cdots + |\alpha|_{mm} \cdot \eta_m = \gamma_m. \end{aligned} \tag{13}$$

6 Numerical Algorithm

Our results allow us to formulate the following

1. Check the conditions $\| \mathbf{D}^{-1} \| < 1$, $\| \mathbf{A} -_D \mathbf{D} \| < 1$.

2. Compute an initial approximation $\mathbf{x}^{(0)}$ applying formulae (10) or (12)–(13) for the interval algebraic problem $\mathrm{mid}(\mathbf{A}) * \mathbf{x} = \mathbf{b}$.

3. Using $\mathbf{x}^{(0)}$ iterate according to

$$\mathbf{x}^{(k+1)} \ := \ \mathbf{D}^{-1} \times \left(\mathbf{b} -_D (\mathbf{A} -_D \mathbf{D}) \times \mathbf{x}^{(k)} \right), \quad k = 0, 1, \ldots$$

The above algorithm was implemented and tested using an experimental package for directed interval arithmetic [14], some additional discussion on the software implementation is given in [10].

7 Conclusion

From our work we hope it becomes evident that algebraic transformations based on directed interval arithmetic can be successfully used for the formulation of iterative procedures for the solution of the interval algebraic problem

(1), resp. for the analysis of the solution (e. g. with respect to its convergence). Thereby directed interval arithmetic can be used in a way much similar to using real arithmetic.

Directed interval arithmetic is the natural arithmetic for the solution of algebraic equations with interval coefficients, since it is obtained from the arithmetic for normal intervals via algebraic completion [8]. Solving interval algebraic equations using the ordinary "naive" interval arithmetic can be compared to solving real linear algebraic equations when restricting ourselves only to the system of positive real numbers.

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